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ABSTRACT

The Jacchia models are represented in terms of spherical harmonic functions. This representation has the advantages of ease of comparison with theoretical and other observational models and data, mathematical analyticity and relative simplicity. The symmetry properties of the models are emphasized by this representation and some physical characteristics like the increase of the amplitude of the diurnal density variation with decreasing solar activity become more apparent.

A REPRESENTATION OF JACCHIA'S THERMOSPHERIC MODELS IN SPHERICAL HARMONICS

The semi-empiric Jacchia thermospheric models (Jacchia 1965, 1970 and 1971) describe thermospheric densities as derived from satellite drag analysis. The densities are obtained in the models from a semi-empiric temperature profile and the global distribution of exospheric temperature. The exospheric temperatures are model parameters that are not necessarily identical with the true kinetic temperatures.

In the Jacchia models the exospheric temperature depends upon the solar activity F, the average solar activity $\bar{F}$ during several solar rotation periods, the geomagnetic activity index K_p, the day in the year, the local solar time (diurnal variation) and the latitude. The variation of exospheric temperature with local time and with latitude is given by a single mathematical expression; i.e., the distribution of exospheric temperature on the globe where the longitude is measured with respect to local noon.

In Jacchia's model (1971) the exospheric temperature is given (in his notation) by:

$$T_\infty = T_c\,(1 + R\sin^m\theta)\left(1 + R\,\frac{\cos^m\eta - \sin^m\theta}{1 + R\sin^m\theta}\cos^3\left(\frac{\tau}{2}\right)\right) \tag{1}$$

where T_c depends only upon solar and geomagnetic activity.

$$T_c = 379^\circ + 3^\circ\!.24\bar{F} + 1^\circ\!.3\,(F - \bar{F}) + 28^\circ K_p + 0^\circ\!.03\exp(K_p)$$

$$\tau = H + \beta + p\sin(H + \gamma) \qquad (-\pi \le \tau \le \pi)$$

$$\eta = \frac{|\phi - \delta|}{2}$$

$$\theta = \frac{|\phi + \delta|}{2}$$

where ϕ is the latitude, δ the declination of the Sun, H the LST counted from local noon, $\beta = -37°$, $p = 6°$, $m = 2.2$, $R = 0.3$, and $\gamma = 43°$.

This representation of the exospheric temperature includes absolute values; therefore it is not analytic. Especially its derivative does not exist at the poles (Blum and Harris, 1973). Jacchia's expression for the exospheric temperature distribution may be approximated by a sum of spherical harmonic functions. Such an approximation is useful because it allows a direct comparison with thermospheric models derived from other considerations like mass-spectrographic data (Hedin et al., 1973) or a theoretical treatment (Harris and Priester, 1962; Mayr, Harris and Volland, 1973). Furthermore, analytic expressions are handled with greater mathematical ease. The pressure gradients, which drive the thermospheric wind system, can only be calculated for the entire globe if the temperature is an analytic function.

In Jacchia's model the exospheric temperature is given as the product of the global minimum exospheric temperature, T_c, and an expression for the temperature dependence on latitude, local time and the declination of the sun; i.e. the day in the year. Thus

$$T_\infty(F, \bar{F}, K_p, \phi, t_A, t_\ell) = T_c(F, \bar{F}, K_p) \cdot G(\phi, t_A, t_\ell) \tag{2}$$

where t_ℓ is the local time and t_A the day in the year.

We have for the moment excluded the semi-annual variation as it will be treated separately. The above factorization of the exospheric temperature representation allows us to express the exospheric temperature in a simple, though very general, form as a sum of spherical harmonic functions $P_n^m(x)$ where x is the cosine of the colatitude.

$$T_\infty = \bar{T}_\infty \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha_{nm}(t_A)\, P_n^m(x) \cos(m\, t_\ell) + \beta_{nm}(t_A)\, P_n^m(x) \sin(m\, t_\ell) \right) \qquad (3)$$

For the numerical results of this paper we have used the usual normalization of the spherical harmonic functions (Jahnke-Emde, p. 111). $\bar{T}_\infty$ is the mean global exospheric temperature that depends on F, $\bar{F}$ and K_p.

Expression (3) can be further simplified by making use of the following properties of Jacchia's model:

(a) The phase of the diurnal variation is independent of latitude and season. Therefore, the cosine and sine terms of the various modes (i.e. the α and β) can be combined so that all the P_n^m that have the same local time mode, m, have the same phase t_m.

(b) The coefficients α_{mn} and β_{mn} depend only on the day in the year and can be expressed as a sum of Fourier modes with the argument the day in the year. These Fourier expressions become particularly simple if the days are counted from vernal equinox. Then only sine coefficients appear in the symmetric terms (i.e. P_n^m with $n - m = 2k$) and cosine coefficients in the antisymmetric terms ($n - m = 2k + 1$). This property is due to the symmetry of the exospheric temperature distribution at equinox.

(c) The third harmonic of the diurnal variation has an amplitude that is less than 0.02 of the amplitude of the fundamental. This is less than the accuracy of drag data and has no physical significance in the Jacchia model. For this reason we have not included the third harmonic in the representation of the temperatures. Due to the non-linearity of the exponential function the relative amplitude of the third harmonic of the density distribution is slightly enhanced. We have included it in the analysis of the density distribution although its physical significance for the densities is equally doubtful.

4. A numerical analysis shows that no terms P_n^m with $n \gtrless 4$ occur with amplitudes larger than 5.10^{-4} of the amplitude of the mean global term P_0^0. Expression (3) simplifies therefore to

$$T_\infty = \bar{T}_\infty \left(\sum_{m=0}^{2} \sum_{n=m}^{4} \gamma_{nm}(t_A)\, P_n^m(x) \cos m\,(t_\ell - t_m) \right) \tag{4}$$

with

$$\gamma_{nm}(t_A) = \sum_{k=0}^{2} \left(\gamma_{nm}^{(k,s)} \cos k\,t_A + \gamma_{nm}^{(k,a)} \sin k\,t_A \right) \tag{5}$$

and $\gamma_{00}^{(0,s)}$ normalized to unity.

The analysis also shows that no higher harmonics than the semi-annual appear in the Fourier analysis of the coefficients $\gamma_{nm}(t_A)$. It should be noted that the semi-annual terms that appear in (5) arise from the variation of the declination of the sun and are quite small. These semi-annual terms should not be confused with the main semi-annual terms that are associated with the semi-annual effect.

Table 1 shows the numerical values of the various terms when the yearly average global term P_0^0 is normalized to unity. Thus Table 1 is independent of solar activity. The relationship between the global minimum exospheric temperature T_c in the Jacchia model and the global mean exospheric temperature used here is given by

$$\bar{T}_\infty = 1.1275\ T_c$$

A synthesis of the exospheric temperatures based on Table 1 deviates less than 1% from Jaachia's original representation. Considering the accuracy of drag data, no loss of physical significance results from the use of this representation as compared to the original model.

THE SEMI-ANNUAL EFFECT AND THE GEOMAGNETIC ACTIVITY EFFECT

In the Jacchia models the semi-annual and the geomagnetic activity correction to the exospheric temperature have no latitudinal dependence. They modify only the P_0^0 terms after the yearly average global term P_0^0, is normalized to unity. As the geomagnetic activity effect has no periodic time dependence, it is simply included in the global average $\bar{T}_\infty$ term and so causes no change in the numerical values of the coefficients given in Table 1. It should be noted that other empirical models of the thermosphere (Hedin et al., 1973) have deduced a latitudinal dependence of the geomagnetic activity effect.

In the 1965 and 1970 Jacchia models the semi-annual variation was expressed as a correction to the exospheric temperature. We have Fourier analysed this correction and obtained

$$T_{sa} = \bar{F}\left(0.2203 \cdot \cos(t_A - 283.7\text{ d}) + 0.3309 \cdot \cos 2(t_A - 30\text{ d}) + 0.0762 \cos 3(t_A - 10.3\text{ d})\right) \qquad (6)$$

The mean annual value of T_{sa} is insignificant and has been neglected. The third annual harmonic has little physical significance, but we have included it in the semi-annual modification given in Table 1. The semi-annual temperature amplitude is directly proportional to the solar activity $\bar{F}$, but the global mean exospheric temperature is only linearly dependent on $\bar{F}$. To generalize the semi-annual correction given in Table 1 for various values of solar activity we have to multiply the semi-annual amplitudes by a function of $\bar{F}$. This function is approximated by F_{sa}:

$$F_{sa} = 1 + 0.296\left(\frac{F - 150}{100}\right) - 0.124\left(\frac{F - 150}{100}\right)^2 \qquad (7)$$

In the Jacchia 1971 model the semi-annual variation is given in terms of densities and not temperatures; therefore it cannot immediately be included in a modification of Table 1.

REPRESENTATION OF DENSITIES

The thermospheric densities are obtained in the Jacchia models from the temperatures by adopting lower boundary densities and assuming diffusive equilibrium. Due to the non-linearity of the exponential function the relative amplitudes of the various spherical harmonic functions are altitude-dependent. There is also a dependence on solar activity. For this reason the direct representation of the densities as a sum of spherical harmonics is less useful then the representation of the exospheric temperature because even the normalized coefficients depend on two parameters: altitude and solar activity. We have given the density representation at a solar activity of $F = 150$ for the altitudes 300, 400 and 500 km in Tables 2-4. These tables do not include the global semi-annual variation. Some spherical harmonic functions that did not appear in the analysis of the temperature, because their normalized amplitude was less than $5 \cdot 10^{-4}$ (the threshhold we have chosen), will have larger amplitudes in the analysis of the densities due to the non-linearities and appear in the Tables 2-4.

The decomposition of the global semi-annual density variation $\rho_{SA}(t_A)$ into Fourier modes according to the 1971 Jacchia model was given by Volland, Wulf-Mathies and Priester (1972). Our independent analysis gave almost identical results. Following Volland et al. the density correction at a height of 400 km is given by

$$\frac{\rho_{SA}(t_y)}{\rho} = 0.1034 \cos (t_y - 4\,d) + 0.1999 \cos 2 (t_y - 109\,d) + 0.0394 \cos 3 (t_y - 66\,d) + 0.0188 \cos 4 (t_y - 14\,d) \tag{8}$$

where t_y is the day in the year counted from January 1.

Our results are

$$\begin{aligned} \frac{\rho_{SA}(t_S)}{\rho} &= 0.1048 \cos (t_A - 288\,d) + 0.2025 \cos 2 (t_A - 28\,d) \\ &\quad + 0.0399 \cos 3 (t_A - 107\,d) + 0.0191 \cos 4 (t_A - 24.6\,d) \\ &= 0.0255 \cos t_A - 0.1016 \sin t_A + 0.1155 \cos 2\,t_A + 0.1664 \sin 2\,t_A \\ &\quad + 0.0290 \cos 3\,t_A - 0.0275 \sin 3\,t_A - 0.0024 \cos 4\,t_A + 0.0189 \sin 4\,t_A \end{aligned} \tag{9}$$

Table 5 gives the density representation including the global semi-annual effect at a height of 400 km and a solar activity of F = 150.

The representation of the drag data in terms of spherical harmonics given in this paper has the advantage of being analytic and relatively simple; i.e. using a small number of constants. It is transparent both from a mathematical and physical point of view and allows the evaluation of the physical significance of some properties of drag data results.

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Table 1

Exospheric Temperature Distribution of the Jacchia Model
(Independent of Solar Activity)

(Excluding Global Semi-Annual Effect)
Coefficients of Spherical Harmonics

Spherical Harmonic	Yearly Average	Annual		Semi-Annual		Third Annual	
	$\cos(0)$	$\cos t_A$	$\sin t_A$	$\cos 2t_A$	$\sin 2t_A$	$\cos 3t_A$	$\sin 3t_A$
P_0^0	1.00						
P_1^0			.0528				
P_2^0	.0054			-.0007			
P_3^0			.0022				
P_4^0	.0041			.0006			
P_5^0							
$P_1^1 \cos(t_\ell - t_1)$	.1300			.0056			
$P_2^1 \cos(t_\ell - t_1)$			.0022				
$P_2^2 \cos 2(t_\ell - t_2)$	.0077						
To include global semi-annual effect replace P_0^0 line by:							
P_0^0	1.00	$.0056\ F_{SA}$	$-.0326\ F_{SA}$	$.0249\ F_{SA}$	$.0427\ F_{SA}$	$.0064\ F_{SA}$	$-.0095\ F_{SA}$

$t_1 = 14.4$

$t_2 = 1.89$

t_A is measured in days from the vernal equinox

t_ℓ is measured in hours from midnight

Table 2

Density Distribution of the Jacchia Model

Height = 300 km $\bar{F}$ = 150

(Excluding Global Semi-Annual Effect)

Coefficients of Spherical Harmonics

Spherical Harmonic	Yearly Average	Annual		Semi-Annual		Third Annual	
	cos (0)	cos t_A	sin t_A	cos $2t_A$	sin $2t_A$	cos $3t_A$	sin $3t_A$
P_0^0	1.0			-.0013			
P_1^0			.1416				.0009
P_2^0	.0126			-.0025			
P_3^0			.0074				
P_4^0	.0112			.0015			
P_5^0							
$P_1^1 \cos (t_\ell - t_1)$	.3492			.0150			
$P_2^1 \cos (t_\ell - t_1)$			.0081				
$P_3^1 \cos (t_\ell - t_1)$	.0013			.0008			
$P_2^2 \cos 2 (t_\ell - t_2)$	.0217			.0009			
$P_3^2 \cos 2 (t_\ell - t_2)$							
$P_4^2 \cos 2 (t_\ell - t_2)$	.0012						
$P_5^2 \cos 2 (t_\ell - t_2)$							
$P_3^3 \cos 3 (t - t_3)$	.0007						

$t_1 = 14.40$ $t_2 = 1.91$ $t_3 = 3.5$

t_A is measured in days from the vernal equinox t_ℓ is measured in hours from midnight

Table 3

Density Distribution of the Jacchia Model

Height = 400 km $\bar{F}$ = 150

(Excluding Global Semi-Annual Effect)

Coefficients of Spherical Harmonics

Spherical Harmonic	Yearly Average	Annual		Semi-Annual		Third Annual	
	cos (0)	cos t_A	sin t_A	cos $2t_A$	sin $2t_A$	cos $3t_A$	sin $3t_A$
P_0^0	1.000			-.0018			
P_1^0			.2091				.0013
P_2^0	.0045			-.0078			
P_3^0			.0104				.0007
P_4^0	.0163			.0018			
P_5^0							
$P_1^1 \cos (t_\ell - t_1)$	.5173			.0219			
$P_2^1 \cos (t_\ell - t_1)$			.0250				.0007
$P_3^1 \cos (t_\ell - t_1)$	.0019			.0007			
$P_2^2 \cos 2 (t_\ell - t_2)$	.0396			.0021			
$P_3^2 \cos 2 (t_\ell - t_2)$			.0010				
$P_4^2 \cos 2 (t_\ell - t_2)$	.0018						
$P_5^2 \cos 2 (t_\ell - t_2)$							
$P_3^3 \cos 3 (t_\ell - t_3)$	.0009						

t_1 = 14.39 t_2 = 2.00 t_3 = 4.4

t_A is measured in days from the vernal equinox t_ℓ is measured in hours from midnight

Table 4

Density Distribution of the Jacchia Model

Height = 500 km $\bar{\bar{F}}$ = 150

(Excluding Global Semi-Annual Effect)

Coefficients of Spherical Harmonics

Spherical Harmonic	Yearly Average cos (0)	Annual cos t_A	Annual sin t_A	Semi-Annual cos $2t_A$	Semi-Annual sin $2t_A$	Third Annual cos $3t_A$	Third Annual sin $3t_A$
P_0^0	1.000			-.0022			
P_1^0			.2726				.0016
P_2^0	-.0104			-.0149			
P_3^0			.0121				.0006
P_4^0	.0211			.0018			
P_5^0			.0017				
$P_1^1 \cos (t_\ell - t_1)$	.6759			.0284			
$P_2^1 \cos (t_\ell - t_1)$			.0476				.0014
$P_3^1 \cos (t_\ell - t_1)$	.0021						
$P_2^2 \cos 2 (t_\ell - t_2)$	.0604			.0034			
$P_3^2 \cos 2 (t_\ell - t_2)$			.0025				
$P_4^2 \cos 2 (t_\ell - t_2)$	.0024						
$P_3^3 \cos 3 (t_\ell - t_3)$	.0016						

$t_1 = 14.37$ $t_2 = 2.05$ $t_3 = 5.1$

t_A is measured in days from the vernal equinox t_ℓ is measured in hours from midnight

Table 5

Density Distribution of the Jacchia Model

Height = 400 km $\bar{F}$ = 150

(Including Global Semi-Annual Effect)

Coefficients of Spherical Harmonics

Spherical Harmonic	Yearly Average	Annual		Semi-Annual		Third Annual	
	cos (0)	cos t_A	sin t_A	cos $2t_A$	sin $2t_A$	cos $3t_A$	sin $3t_A$
P_0^0	1.000	.0251	-.1004	.1122	.1642	.0286	-.0271
P_1^0	-.0106	.0175	.1973	.0075		-.0152	.0135
P_2^0	.0040		-.0007	-.0073	.0007		
P_3^0		.0009	.0099			-.0008	.0013
P_4^0	.0164	.0005	-.0016	.0036	.0027	.0005	-.0005
P_5^0							
$P_1^1 \cos (t_\ell - t_1)$	.5186	.0136	-.0511	.0808	.0851	.0151	-.0150
$P_2^1 \cos (t_\ell - t_1)$	-.0013	.0021	.0237	.0009		-.0018	.0022
$P_3^1 \cos (t_\ell - t_1)$	.0019			.0009			
$P_2^2 \cos 2 (t_\ell - t_2)$	.0397	.0010	-.0039	.0065	.0065	.0012	-.0012
$P_3^2 \cos 2 (t_\ell - t_2)$			.0010				
$P_4^2 \cos 2 (t_\ell - t_2)$	.0018						
$P_3^3 \cos 3 (t_\ell - t_3)$	.0009						

t_1 = 14.40 t_2 = 2.00 t_3 = 4.4

t_A is measured in days from the vernal equinox t_ℓ is measured in hours from midnight